Landau Damping, the Caldeira-Leggett Model, and the Linearized Vlasov-Poisson System
Sherwood Meeting 2011

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Overview: Diagonalizing Hamiltonian Systems with Continuous Spectra

- Exactly solving linear systems with continuous spectra.

- Introduce the Caldeira-Leggett model:
  - Linear Hamiltonian system used to describe dissipation in quantum mechanics.

- Derive a transformation which diagonalizes this system. Exact solution.

- Use transformation to demonstrate equivalence to linearized Vlasov-Poisson equation.

- Application: Prediction of echo effect in these systems.
Caldeira-Leggett Model: The Standard Model for a Dissipative Quantum System

- Axioms of quantum mechanics require symmetric Hamiltonian operators $H$, and $i\hbar \frac{\partial \psi}{\partial t} = H\psi$. Typical dissipative operators are not of this form.
- Caldeira-Leggett model was invented to include dissipation in quantum mechanics through continuum damping.
- Let the system of interest have Hamiltonian $H_s$. Dissipation occurs through some environment. Explicitly model the environment with $H_e$ and coupling $H_c$.
- Total Hamiltonian is $H = H_s + H_c + H_e$.
- Infinite number of degrees of freedom can lead to a continuous spectrum and damping of the system of interest.
- Caldeira-Leggett model assumes the environment is a bath of oscillators with a linear coupling to the system.
Caldeira-Leggett Hamiltonian

- Consider the classical system with $H_s$ of a simple harmonic oscillator, $H_e$ of a continuous bath of oscillators, and $H_c$ a simple linear coupling:

\[
H_{CL}[q, p; Q, P] = \frac{\Omega}{2} P^2 + \frac{1}{2} \left( \Omega + \int_{\mathbb{R}_+} dx \frac{f(x)^2}{2x} \right) Q^2 \\
+ \int_{\mathbb{R}_+} dx \left[ \frac{x}{2} (p(x)^2 + q(x)^2) + Qq(x) f(x) \right],
\]

\[
\dot{q}(x) = xp(x) \\
\dot{p}(x) = -xq(x) - Qf(x) \\
\dot{Q} = \Omega P \\
\dot{P} = -\left( \Omega + \int_{\mathbb{R}_+} dx \frac{f(x)^2}{2x} \right) Q - \int_{\mathbb{R}_+} dx q(x)f(x).
\]
Caldeira-Leggett Model Illustration

Dashed Lines Indicate Couplings. All Motion Confined to the Vertical Direction.

Representation of a Continuum of Bath Oscillators
Caldeira-Leggett Example: Resistively Shunted Josephson Junction and Luttinger Liquid

- Caldeira-Leggett has been used to describe tunnelling in the presence of dissipation after modification of system potential. Resistively shunted Josephson junctions.
- Bosonic excitations of a Luttinger liquid have also been treated this way.

Figure: Circuit Diagram for Resistively Shunted Josephson Junction
Cold-Trapped Ions as Realizations of Caldeira-Leggett

- Ions can be trapped in a Paul trap using the ponderomotive force in various types of cavities.
- Harmonic potential and interaction with potentially noisy environment.

**Figure:** Apparatus for trapping an ion using r.f. fields. From Myatt et. al. *Nature.* 269-273. 2000
Solving the Caldeira-Leggett Model: Analogue of Van-Kampen Modes

▶ Guess solutions like $\sim e^{-iut}$.

▶ Typically there are no discrete eigenvalues.

▶ Instead there are continuum eigenmodes, for each real $u$:

$$q_u(x) = \text{PV} \frac{Q_uf(x)}{u^2 - x^2} + C_u Q_u \delta(|u| - x).$$

$$C_u = \frac{u^2 - \Omega^2_c}{\Omega f(|u|)} - \int_{\mathbb{R}} dx \frac{f(|x|)^2}{2(u - x)f(|u|)}.$$

▶ Here $Q_u$ is an arbitrary amplitude. The other continuum eigenmodes can be formally solved for using the eigenvalue equations.
Partial Solution of Caldeira-Leggett Through Eigenfunctions

- The amplitudes $\frac{\partial Q_u}{\partial t} = -iut Q_u$. Gives a solution to Caldeira-Leggett.

$$q(x, t) = \int_\mathbb{R} du \frac{Q_u x f(x)}{u^2 - x^2} e^{-iut} + \int_\mathbb{R} du C_u Q_u \delta(|u| - x) e^{-iut}$$

$$Q(t) = \int_\mathbb{R} du Q_u e^{-iut},$$

$$p(x, t) = -\int_\mathbb{R} du \frac{iQ_u x f(x)}{u^2 - x^2} e^{-iut} - \int_\mathbb{R} du \frac{iu}{x} C_u Q_u \delta(|u| - x) e^{-iut}$$

$$P(t) = -\int_\mathbb{R} du \frac{iu}{\Omega} Q_u e^{-iut},$$
Transformation from $Q_u$ can be Written Using Singular Integral Operators

- Make the definitions:

$$H[g](v) = \frac{1}{\pi} \int_{\mathbb{R}} dx \frac{g(x)}{x - v}.$$ 

$$\epsilon_I = \pi f(x)^2 \text{sgn}(x) \quad \text{and} \quad \epsilon_R = 2\frac{x^2 - \Omega_c^2}{\Omega} + \pi H[f(|x|)^2].$$

$$T_+[h](u) := \epsilon_R h(|u|) + \epsilon_I H[h(|x|)](u),$$

$$T_-[h](u) := \epsilon_R h(|u|) + \epsilon_I H[\text{sgn}(x)h(|x|)](u).$$

- Then the previous transformation from $Q_u$ to $(q(x), Q)$ can be written in terms of the symmetric part of $Q_u$, which is called $Q_{u+}$:

$$l_c[Q+] := \left(\frac{1}{f(x)} T_+[Q_{u+}], 2 \int_{\mathbb{R}_+} du Q_{u+}\right).$$
Let $h(t)$ be a function defined on $\mathbb{R}$. Then the define $\phi(z)$ by the Cauchy integral:

$$\phi(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(t)dt}{t - z}$$

This function is sectionally analytic in the lower and upper half plane. At the real axis there is a jump discontinuity:

$$\phi_+(t) - \phi_-(t) = h(t) \quad \phi_+(t) + \phi_-(t) = -H[h](t)$$

Therefore the function $h(t) + iH[h](t)$ is the boundary value of an analytic function in the upper half plane, and $h(t) - iH[h](t)$ is the boundary value of an analytic function in the lower half plane, for any $h$.

Can write elements of the transformation in this form.
Inverting the Transform Using the Generalized Liouville Theorem

- Must solve for $Q_{u+}$:

$$f(x)q(x) = \epsilon_R Q_{u+} + \epsilon_I H[Q_{u+}]$$

- Subject to the constraints $Q_{u+}$ is symmetric and $2 \int_{\mathbb{R}^+} Q_u du = Q$.

- Replace each term as a sum of an analytic function in the upper half plane and an analytic function in the lower half plane.

- Isolate all the upper half plane terms onto the opposite side of all the lower half plane terms.

- The two sides collectively define an entire function.

- Apply generalized Liouville theorem: An entire function with a growth rate of $z^n$ at infinity is a polynomial of degree $n$.

- Choose the polynomial to satisfy the constraints.
To this write the inverse transformation define:

\[ \hat{T}_+ [h](u) := \frac{\epsilon_R}{|\epsilon|^2} h(u) - \frac{\epsilon_I}{|\epsilon|^2} H[h(|x|)](u), \]

\[ \hat{T}_- [h](u) := \frac{\epsilon_R}{|\epsilon|^2} h(u) - \frac{\epsilon_I}{|\epsilon|^2} H[\text{sgn}(x) h(|x|)](u). \]

\[ Q_{u+} = \hat{I}_c[q(x), Q] = \hat{T}_+[f(x)q(x)] + \frac{2u}{\pi \Omega} \frac{\epsilon_I}{|\epsilon|^2} Q. \]

\[ \bar{Q}(u) = \sqrt{\frac{\pi}{|\epsilon|^2}} \epsilon_I Q_{u+} = \sqrt{\frac{\pi}{|\epsilon|^2}} \epsilon_I \hat{T}_+[f(x)q(x)] + \frac{2u}{\Omega} \sqrt{\frac{|\epsilon_I|}{\pi |\epsilon|^2}} Q. \]

Armed with a transformation from \((q(x), Q)\) to \(\bar{Q}(u)\), we use a canonical transformation to complete the diagonalization of the Caldeira-Leggett model.
Canonical Transformation

Define the type-2 mixed variable generating functional:

\[ \mathcal{F}[q, Q, \bar{P}] = \int_{\mathbb{R}^+} \bar{P} \sqrt{\frac{\pi}{|\epsilon|^2}} \epsilon_l \hat{I}[q(x), Q] \]

Then the rest of the transformation is:

\[ p(x) = \frac{\delta \mathcal{F}}{\delta q} = f(x) \hat{T}_+^\dagger \left[ \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_l}} \bar{P} \right] \]

\[ P = \frac{\partial \mathcal{F}}{\partial Q} = \int_{\mathbb{R}^+} du \frac{2u\bar{P}}{\Omega} \sqrt{\frac{\epsilon_l}{\pi|\epsilon|^2}} \]

\[ \bar{P} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_l}} \left( \hat{T}_- [f(x)p(x)] + \frac{2}{\pi} \frac{\epsilon_l}{|\epsilon|^2} P \right) \]
New Hamiltonian: Pure Continuum of Harmonic Oscillators

- New variables $\bar{P}$ and $\bar{Q}$

\[
\bar{P} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_l}} \left( \hat{T}_- [f(x)p(x)] + \frac{2}{\pi} \frac{\epsilon_l}{|\epsilon|^2} P \right)
\]

\[
\bar{Q} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_l}} \left( \hat{T}_+ [f(x)q(x)] + \frac{2u}{\pi \Omega} \frac{\epsilon_l}{|\epsilon|^2} Q \right).
\]

- On direct substitution, these variables convert the Hamiltonian of the Caldeira-Leggett model into a pure continuum of harmonic oscillators with Hamiltonian:

\[
H[\bar{Q}, \bar{P}] = \int_{\mathbb{R}^+} \frac{du}{2} \left( \bar{Q}(u)^2 + \bar{P}(u)^2 \right)
\]

- These transformations allow the Caldeira-Leggett model to be solved exactly.
Damping in the Caldeira-Leggett Model

- Consider an initial condition in the Caldeira-Leggett model.
- There is a corresponding initial condition $\bar{Q}(u, 0), \bar{P}(u, 0)$.
- The solution in terms of these variables is:

$$
\bar{Q}(u, t) = \bar{Q}(u, 0)\cos(ut) + \bar{P}(u, 0)\sin(ut)
$$

$$
\bar{P}(u, t) = \bar{P}(u, 0)\cos(ut) - \bar{Q}(u, 0)\sin(ut)
$$

- From this,

$$
Q(t) = \int_{\mathbb{R}_+} \frac{2u}{\Omega} \sqrt{\frac{e_l}{\pi |\epsilon|^2}} (\bar{Q}(u, 0)\cos(ut) + \bar{P}(u, 0)\sin(ut))
$$

- By the Riemann-Lebesgue lemma this decays to zero with time.

- Damping is continuum damping.

- Completely analogous calculation can be made to Landau damping of a plasma. The damping here has the nature of Landau damping.
Consider the linearized Vlasov-Poisson equation in Fourier space:

\[
\frac{\partial f_k}{\partial t} - ikvf_k - \frac{4\pi ie^2}{mk} f'_0(v) \int_{\mathbb{R}} dv f_k = 0.
\]

There is a transformation, due to Morrison, that allows the exact solution of the Vlasov-Poisson equation:

\[
\epsilon_I(v) = -\frac{4\pi^2 e^2 f'_0}{mk^2 \int_{\mathbb{R}} dv f_0}, \quad \epsilon_R(v) = 1 + H[\epsilon_I],
\]

\[
G_k[f] = \epsilon_R f + \epsilon_I H[f] \quad \text{and} \quad \hat{G}_k[f] = \frac{\epsilon_R}{|\epsilon|^2} f - \frac{\epsilon_I}{|\epsilon|^2} H[f].
\]

\[
Q_k(u) = \hat{G}_k[f_k] \text{ satisfies } \frac{\partial Q_k(u)}{\partial t} = -iukQ_k(u).
\]

This makes it possible to identify solutions of Caldeira-Leggett with Vlasov-Poisson and vice-versa.
Plasma Echo: An Inspiration for an Application?

- In plasmas the density or electric field is typically measured. How do we know that the phase space structure of the distribution function is preserved?
- Using weakly nonlinear Vlasov theory, two perturbations, separated by time \( t = \tau \) of the electric field that have Landau damped may interact and cause the re-appearance of the electric field at some later time \( t \).
- The plasma echo was predicted and observed in the 1960s.
- The strength of the echo makes it possible to determine the collisionality.
- Could it be possible to observe some type of echo effect in the Caldeira-Leggett model?
Echo Effect in the Caldeira-Leggett Model

- Relying on the linear theory, and echo may be triggered by driving the bath.
- Suppose the bath Hamiltonian depends on a parameter:
  \[ H[q(x), p(x)] = \int \frac{dx}{2} (c(t)q(x)^2 + p(x)^2) \]
- Let \( c(t) = 1 + c\delta(t - \tau) \), which implies that the bath is driven by a strong impulse.
- At \( t = \tau \) the bath coordinates are transformed.
  \[ q(x, \tau) = q(x, \tau), \quad p(\tau) = p(\tau) - cq(\tau). \]
- Result is partial time reversal of the evolution:
  \[
  \frac{\partial q(x, \tau)}{\partial t} = xp(x, \tau) - cxq(x, \tau)
  \]
  \[
  \frac{\partial p(x, \tau)}{\partial t} = -xq(x, \tau) + cx/2(p(x, \tau) - p(x, \tau))
  \]
- At time \( t = 2\tau \) there is an echo and the initial perturbation to \( Q \) is recovered, with magnitude \( c \).
Candidate Systems: Bose-Einstein Condensate

- Caldeira-Leggett is used for a large number of types of systems, with widely varying types of baths.

- One intriguing (highly speculative) possibility is the interaction of a Bose-Einstein condensate with a neutral atom trapped in an optical lattice.

- Confinement of neutrals due to AC Stark effect. Shift of ground state energy. Harmonic potentials possible.

- Bose-Einstein condensate can coexist with confined neutrals.

- Study interaction of a neutral (system) with the bath (excitations of the BEC).

- Drive the Bose-Condensate with a laser to trigger an echo.
Diagram of Optical Lattice

- There are a large number of possible choices for this configuration, including components for the BEC, the neutrals, the optical lattice, and the method of excitation.

Figure: Optical lattice with trapped neutrals. Imposition of Bose-Einstein condensate could lead to realization of the Caldeira-Leggett model through interaction with Bose-Einstein spectrum. Figure from NIST.
Conclusion

- Found a transformation that allows for the exact solution of the Caldeira-Leggett model.

- Use diagonalization to explain the damping in the Caldeira-Leggett model as continuum damping.

- Established an exact equivalence with the linearized Vlasov-Poisson equation.

- Suggested the existence of an echo in the Caldeira-Leggett model.

- Detection and use of echoes could lead to a large variety of interesting experiments in systems described by the Caldeira-Leggett model.