Landau Damping, the Caldeira-Leggett Model, and the Linearized Vlasov-Poisson System
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Overview: Diagonalizing Hamiltonian Systems with Continuous Spectra

- Exactly solving linear systems with continuous spectra.
- Introduce the Caldeira-Leggett model:
  - Linear Hamiltonian system used to describe dissipation in quantum mechanics.
- Derive a transformation which diagonalizes this system. Exact solution.
- Use transformation to demonstrate equivalence to linearized Vlasov-Poisson equation.
- Application: Prediction of echo effect in these systems.

Caldeira-Leggett Model: The Standard Model for a Dissipative Quantum System

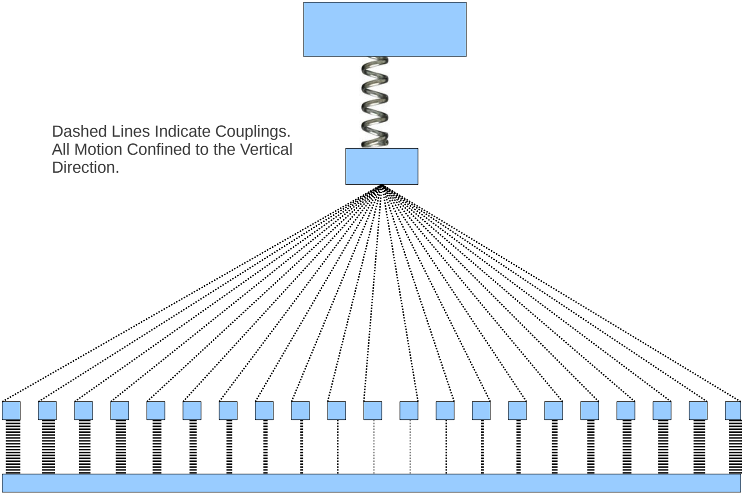
- Axioms of quantum mechanics require symmetric Hamiltonian operators H, and $ih\frac{\partial\psi}{\partial t} = H\psi$. Typical dissipative operators are not of this form.
- Caldeira-Leggett model was invented to include dissipation in quantum mechanics through continuum damping.
- Let the system of interest have Hamiltonian H_s. Dissipation occurs through some environment. Explicitly model the environment with H_e and coupling H_c.
- Total Hamiltonian is $H = H_s + H_c + H_e$.
- Infinite number of degrees of freedom can lead to a continuous spectrum and damping of the system of interest.
- Caldeira-Leggett model assumes the environment is a bath of oscillators with a linear coupling to the system.

Caldeira-Leggett Hamiltonian

- Consider the classical system with H_s of a simple harmonic oscillator, H_e of a continuous bath of oscillators, and H_c a simple linear coupling:

$$\begin{aligned} H_{CL}[q, p; Q, P] &= \frac{\Omega}{2} P^2 + \frac{1}{2}\left(\Omega + \int_{\mathbb{R}_+} dx \, \frac{f(x)^2}{2x}\right) Q^2 \\ &\quad + \int_{\mathbb{R}_+} dx \left[\frac{x}{2}(p(x)^2 + q(x)^2) + Qq(x)f(x)\right], \\ \dot{q}(x) &= xp(x) \\ \dot{p}(x) &= -xq(x) - Qf(x) \\ \dot{Q} &= \Omega P \\ \dot{P} &= -\left(\Omega + \int_{\mathbb{R}_+} dx \, \frac{f(x)^2}{2x}\right) Q - \int_{\mathbb{R}_+} dx \, q(x) f(x) \, . \end{aligned}$$

Caldeira-Leggett Model Illustration



Representation of a Continuum of Bath Oscillators

Caldeira-Leggett Example: Resistively Shunted Josephson Junction and Luttinger Liquid

- Caldeira-Leggett has been used to describe tunnelling in the presence of dissipation after modification of system potential. Resistively shunted Josephson junctions.
- Bosonic excitations of a Luttinger liquid have also been treated this way.

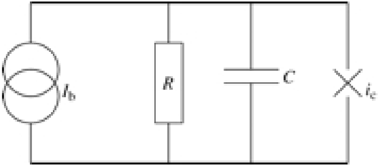


Figure: Circuit Diagram for Resistively Shunted Josephson Junction

Cold-Trapped Ions as Realizations of Caldeira-Leggett

- Ions can be trapped in a Paul trap using the ponderomotive force in arious types of cavities.
- Harmonic potential and interaction with potentially noisy environment.

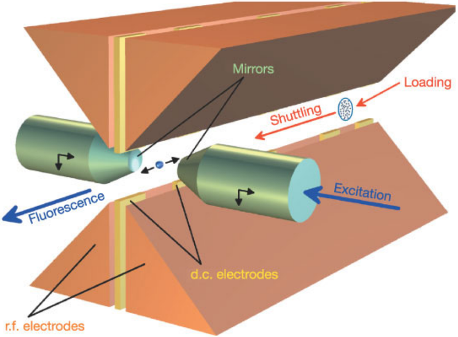


Figure: Apparatus for trapping an ion using r.f. fields. From Myatt et. al. **Nature**. 269-273. 2000

Solving the Caldeira-Leggett Model: Analogue of Van-Kampen Modes

- Guess solutions like $\sim e^{-iut}$.

- Typically there are no discrete eigenvalues.

- Instead there are continuum eigenmodes, for each real u:

$$q_u(x) = \mathbf{PV}\,\frac{Q_u x f(x)}{u^2 - x^2} + C_u Q_u \delta(|u| - x)\,.$$

$$C_u = \frac{u^2 - \Omega_c^2}{\Omega f(|u|)} - \int_{\mathbb{R}} dx\, \frac{f(|x|)^2_-}{2(u-x) f(|u|)}\,.$$

- Here Q_u is an arbitrary amplitude. The other continuum eigenmodes can be formally solved for using the eigenvalue equations.

Partial Solution of Caldeira-Leggett Through Eigenfunctions

- The amplitudes $\frac{\partial Q_u}{\partial t} = -iutQ_u$. Gives a solution to Caldeira-Leggett.

$$
\begin{aligned}
q(x,t) &= \fint_{\mathbb{R}} du\, \frac{Q_u x f(x)}{u^2 - x^2} e^{-iut} + \int_{\mathbb{R}} du\, C_u Q_u \delta(|u| - x) e^{-iut} \\
Q(t) &= \int_{\mathbb{R}} du\, Q_u e^{-iut}\,, \\
p(x,t) &= -\fint_{\mathbb{R}} du\, \frac{iQ_u u f(x)}{u^2 - x^2} e^{-iut} - \int_{\mathbb{R}} du\, \frac{iu}{x} C_u Q_u \delta(|u| - x) e^{-iut} \\
P(t) &= -\int_{\mathbb{R}} du\, \frac{iu}{\Omega} Q_u e^{-iut}\,,
\end{aligned}
$$

Transformation from Q_u can be Written Using Singular Integral Operators

- Make the definitions:

$$H[g](v) = \frac{1}{\pi}\int_{\mathbb{R}} dx\, \frac{g(x)}{x-v}\,.$$

$$\epsilon_I = \pi f(x)^2 sgn(x) \qquad \text{and}$$

$$\epsilon_R = 2\frac{x^2-\Omega_c^2}{\Omega} + \pi H[f(|x|)_-^2]\,.$$

$$T_+[h](u) := \epsilon_R h(|u|) + \epsilon_I H[h(|x|)](u)\,,$$

$$T_-[h](u) := \epsilon_R h(|u|) + \epsilon_I H[sgn(x)h(|x|)](u)\,,$$

- Then the previous transformation from Q_u to $(q(x), Q)$ can be written in terms of the symmetric part of Q_u, which is called Q_{u+}:

$$I_c[Q_+] := \left(\frac{1}{f(x)}T_+[Q_{u+}], 2\int_{\mathbb{R}_+} du\, Q_{u+}\right)$$

Singular Integral Equations and Riemann-Hilbert Problems

- Let $h(t)$ be a function defined on $\mathbb{R}$. Then the define $\phi(z)$ by the Cauchy integral:

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{h(t)dt}{t - z}$$

- This function is sectionally analytic in the lower and upper half plane. At the real axis there is a jump discontinuity:

$$\phi_+(t) - \phi_-(t) = h(t) \qquad \phi_+(t) + \phi_-(t) = -H[h](t)$$

- Therefore the function $h(t) + iH[h](t)$ is the boundary value of an analytic function in the upper half plane, and $h(t) - iH[h](t)$ is the boundary value of an analytic function in the lower half plane, for any h.
- Can write elements of the transformation in this form.

Inverting the Transform Using the Generalized Liouville Theorem

- Must solve for Q_{u+}:

$$f(x)q(x) = \epsilon_R Q_{u+} + \epsilon_I H[Q_{u+}]$$

- Subject to the constraints Q_{u+} is symmetric and $2\int_{\mathbb{R}_+} Q_u du = Q$.
- Replace each term as a sum of an analytic function in the upper half plane and an analytic function in the lower half plane.
- Isolate all the upper half plane terms onto the opposite side of all the lower half plane terms.
- The two sides collectively define an entire function.
- Apply generalized Liouville theorem: An entire function with a growth rate of z^n at infinity is a polynomial of degree n.
- Choose the polynomial to satisfy the constraints.

Inverting the Transform II

- To this write the inverse transformation define:

$$\widehat{T_+}[h](u) := \frac{\epsilon_R}{|\epsilon|^2} h(u) - \frac{\epsilon_I}{|\epsilon|^2} H[h(|x|)](u)\,,$$

$$\widehat{T_-}[h](u) := \frac{\epsilon_R}{|\epsilon|^2} h(u) - \frac{\epsilon_I}{|\epsilon|^2} H[sgn(x)h(|x|)](u)\,.$$

$$Q_{u+} = \widehat{I_c}[q(x), Q] = \widehat{T_+}[f(x)q(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2} Q\,.$$

$$\bar{Q}(u) = \sqrt{\frac{\pi}{|\epsilon|^2}\epsilon_I} Q_{u+} = \sqrt{\frac{\pi}{|\epsilon|^2}\epsilon_I}\,\widehat{T_+}[f(x)q(x)] + \frac{2u}{\Omega}\sqrt{\frac{|\epsilon_I|}{\pi|\epsilon|^2}} Q$$

- Armed with a transformation from $(q(x), Q)$ to $\bar{Q}(u)$, we use a canonical transformation to complete the diagonalization of the Caldeira-Leggett model.

Canonical Transformation

- Define the type-2 mixed variable generating functional:

$$\mathcal{F}[q, Q, \bar{P}] = \int_{\mathbb{R}_+} \bar{P} \sqrt{\frac{\pi}{|\epsilon|^2} \epsilon_I} \widehat{I}[q(x), Q]$$

- Then the rest of the transformation is:

$$p(x) = \frac{\delta \mathcal{F}}{\delta q} = f(x) \hat{T}_+^\dagger \left[\sqrt{\frac{\pi |\epsilon^2|}{\epsilon_I}} \bar{P} \right]$$

$$P = \frac{\partial \mathcal{F}}{\partial Q} = \int_{R_+} du \frac{2u\bar{P}}{\Omega} \sqrt{\frac{\epsilon_I}{\pi |\epsilon|^2}}$$

$$\bar{P} = \sqrt{\frac{\pi |\epsilon|^2}{\epsilon_I}} \left(\widehat{T_-}[f(x)p(x)] + \frac{2}{\pi} \frac{\epsilon_I}{|\epsilon|^2} P \right)$$

New Hamiltonian: Pure Continuum of Harmonic Oscillators

- New variables $\bar{P}$ and $\bar{Q}$

$$\bar{P} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_-}[f(x)p(x)] + \frac{2}{\pi}\frac{\epsilon_I}{|\epsilon|^2}P\right)$$

$$\bar{Q} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}}\left(\widehat{T_+}[f(x)q(x)] + \frac{2u}{\pi\Omega}\frac{\epsilon_I}{|\epsilon|^2}Q\right) .$$

- On direct substitution, these variables convert the Hamiltonian of the Caldeira-Leggett model into a pure continuum of harmonic oscillators with Hamiltonian:

$$H[\bar{Q}, \bar{P}] = \int_{\mathbb{R}_+} \frac{du}{2}\left(\bar{Q}(u)^2 + \bar{P}(u)^2\right)$$

- These transformations allow the Caldeira-Leggett model to be solved exactly.

Damping in the Caldeira-Leggett Model

- Consider an initial condition in the Caldeira-Leggett model.
- There is a corresponding initial condition $\bar{Q}(u,0)$, $\bar{P}(u,0)$.
- The solution in terms of these variables is:

$$\bar{Q}(u,t) = \bar{Q}(u,0)cos(ut) + \bar{P}(u,0)sin(ut)$$
$$\bar{P}(u,t) = \bar{P}(u,0)cos(ut) - \bar{Q}(u,0)sin(ut)$$

- From this, $Q(t) = \int_{\mathbb{R}_+} \frac{2u}{\Omega}\sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}}(\bar{Q}(u,0)cos(ut) + \bar{P}(u,0)sin(ut))$.
- By the Riemann-Lebesgue lemma this decays to zero with time.
- Damping is continuum damping.
- Completely analogous calculation can be made to Landau damping of a plasma. The damping here has the nature of Landau damping.

Linearized Vlasov-Poisson: A Similar Situation

- Consider the linearized Vlasov-Poisson equation in Fourier space:
$$\frac{\partial f_k}{\partial t} - ikvf_k - \frac{4\pi i e^2}{mk} f_0'(v) \int_{\mathbb{R}} dv\, f_k = 0\,.$$
- There is a transformation, due to Morrison, that allows the exact solution of the Vlasov-Poisson equation:

$$\varepsilon_I(v) = -\frac{4\pi^2 e^2 f_0'}{mk^2 \int_{\mathbb{R}} dv\, f_0} \qquad \varepsilon_R(v) = 1 + H[\varepsilon_I]\,,$$

$$G_k[f] = \varepsilon_R f + \varepsilon_I H[f] \qquad \widehat{G_k}[f] = \frac{\varepsilon_R}{|\varepsilon|^2} f - \frac{\varepsilon_I}{|\varepsilon|^2} H[f]\,.$$

- $Q_k(u) = \hat{G}_k[f_k]$ satisfies $\frac{\partial Q_k(u)}{\partial t} = -iukQ_k(u)$.
- This makes it possible to identify solutions of Caldeira-Leggett with Vlasov-Poisson and vice-versa.

Plasma Echo: An Inspiration for an Application?

- In plasmas the density or electric field is typically measured. How do we know that the phase space structure of the distribution function is preserved?
- Using weakly nonlinear Vlasov theory, two perturbations, separated by time $t = \tau$ of the electric field that have Landau damped may interact and cause the re-appearance of the electric field at some later time t.
- The plasma echo was predicted and observed in the 1960s.
- The strength of the echo makes it possible to determine the collisionality.
- Could it be possible to observe some type of echo effect in the Caldeira-Leggett model?
- This echo has been suggested in an unpublished arXiv submission by Kuklov, **arXiv:cond-mat/9803351v1** (1998).

Echo Effect in the Caldeira-Leggett Model

- Relying on the linear theory, and echo may be triggered by driving the bath.
- Suppose the bath Hamiltonian depends on a parameter:

$$H[q(x), p(x)] = \int \frac{dx}{2} \left(c(t)q(x)^2 + p(x)^2 \right)$$

- Let $c(t) = 1 + c\delta(t - \tau)$, which implies that the bath is driven by a strong impulse.
- At $t = \tau$ the bath coordinates are transformed. $q(x, \tau) = q(x, \tau)$, $p(\tau) = p(\tau) - cq(\tau)$.
- Result is partial time reversal of the evolution:

$$\frac{\partial q(x, \tau)}{\partial t} = xp(x, \tau) - cxq(x, \tau)$$
$$\frac{\partial p(x, \tau)}{\partial t} = -xq(x, \tau) + cx/2(p(x, \tau) - p(x, \tau))$$

- At time $t = 2\tau$ there is an echo and the initial perturbation to Q is recovered, with magnitude c.

Candidate Systems: Bose-Einstein Condensate

- Caldeira-Leggett is used for a large number of types of systems, with widely varying types of baths.
- One intriguing (highly speculative) possibility is the interaction of a Bose-Einstein condensate with a neutral atom trapped in an optical lattice.
- Confinement of neutrals due to AC Stark effect. Shift of ground state energy. Harmonic potentials possible.
- Bose-Einstein condensate can coexist with confined neutrals.
- Study interaction of a neutral (system) with the bath (excitations of the BEC).
- Drive the Bose-Condensate with a laser to trigger an echo.

Diagram of Optical Lattice

- There are a large number of possible choices for this configuration, including components for the BEC, the neutrals, the optical lattice, and the method of excitation.

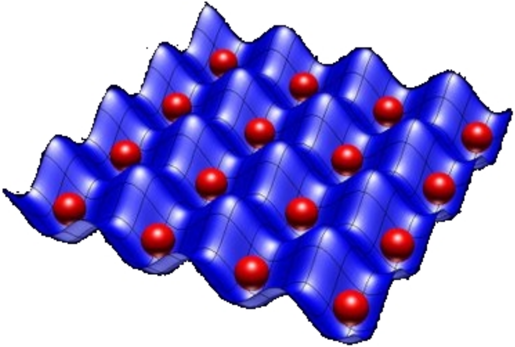


Figure: Optical lattice with trapped neturals. Imposition of Bose-Einstein condensate could lead to realization of the Caldeira-Leggett model through interaction with Bose-Einstein spectrum. Figure from NIST.

Conclusion

- Found a transformation that allows for the exact solution of the Caldeira-Leggett model.
- Use diagonalization to explain the damping in the Caldeira-Leggett model as continuum damping.
- Established an exact equivalence with the linearized Vlasov-Poisson equation.
- Suggested the existence of an echo in the Caldeira-Leggett model.
- Detection and use of echos could lead to a large variety of interesting experiments in systems described by the Caldeira-Leggett model.