

Landau Damping, the Caldeira-Leggett Model, and the Linearized Vlasov-Poisson System

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Overview: Diagonalizing Hamiltonian Systems with Continuous Spectra

- ▶ Exactly solving linear systems with continuous spectra.
- ▶ Introduce the Caldeira-Leggett model:
 - ▶ Linear Hamiltonian system used to describe dissipation in quantum mechanics.
- ▶ Derive a transformation which diagonalizes this system. Exact solution.
- ▶ Use transformation to demonstrate equivalence to linearized Vlasov-Poisson equation.
- ▶ Application: Prediction of echo effect in these systems.

Caldeira-Leggett Model: The Standard Model for a Dissipative Quantum System

- ▶ Axioms of quantum mechanics require symmetric Hamiltonian operators H , and $ih\frac{\partial\psi}{\partial t} = H\psi$. Typical dissipative operators are not of this form.
- ▶ Caldeira-Leggett model was invented to include dissipation in quantum mechanics through continuum damping.
- ▶ Let the system of interest have Hamiltonian H_s . Dissipation occurs through some environment. Explicitly model the environment with H_e and coupling H_c .
- ▶ Total Hamiltonian is $H = H_s + H_c + H_e$.
- ▶ Infinite number of degrees of freedom can lead to a continuous spectrum and damping of the system of interest.
- ▶ Caldeira-Leggett model assumes the environment is a bath of oscillators with a linear coupling to the system.

Caldeira-Leggett Hamiltonian

- ▶ Consider the classical system with H_s of a simple harmonic oscillator, H_e of a continuous bath of oscillators, and H_c a simple linear coupling:

$$H_{CL}[q, p; Q, P] = \frac{\Omega}{2} P^2 + \frac{1}{2} \left(\Omega + \int_{\mathbb{R}_+} dx \frac{f(x)^2}{2x} \right) Q^2 \\ + \int_{\mathbb{R}_+} dx \left[\frac{x}{2} (p(x)^2 + q(x)^2) + Qq(x)f(x) \right],$$

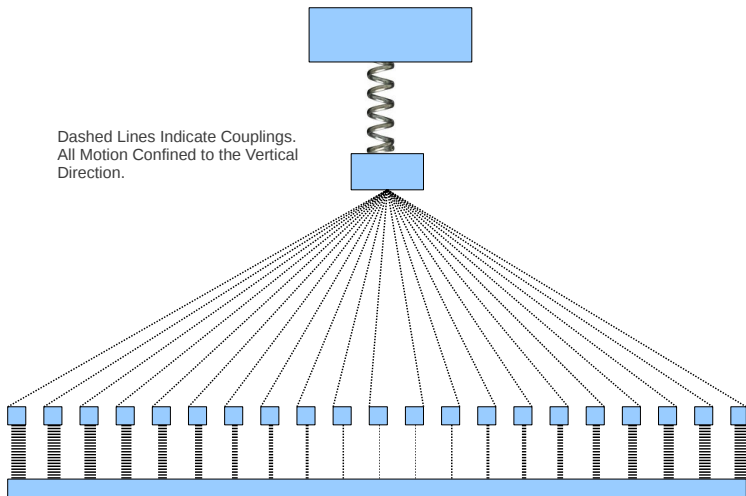
$$\dot{q}(x) = xp(x)$$

$$\dot{p}(x) = -xq(x) - Qf(x)$$

$$\dot{Q} = \Omega P$$

$$\dot{P} = - \left(\Omega + \int_{\mathbb{R}_+} dx \frac{f(x)^2}{2x} \right) Q - \int_{\mathbb{R}_+} dx q(x)f(x).$$

Caldeira-Leggett Model Illustration



Representation of a Continuum of Bath Oscillators

Caldeira-Leggett Example: Resistively Shunted Josephson Junction and Luttinger Liquid

- ▶ Caldeira-Leggett has been used to describe tunnelling in the presence of dissipation after modification of system potential. Resistively shunted Josephson junctions.
- ▶ Bosonic excitations of a Luttinger liquid have also been treated this way.

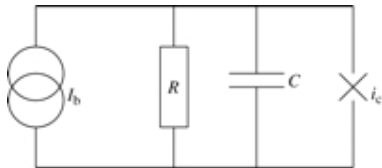


Figure: Circuit Diagram for Resistively Shunted Josephson Junction

Cold-Trapped Ions as Realizations of Caldeira-Leggett

- ▶ Ions can be trapped in a Paul trap using the ponderomotive force in various types of cavities.
- ▶ Harmonic potential and interaction with potentially noisy environment.

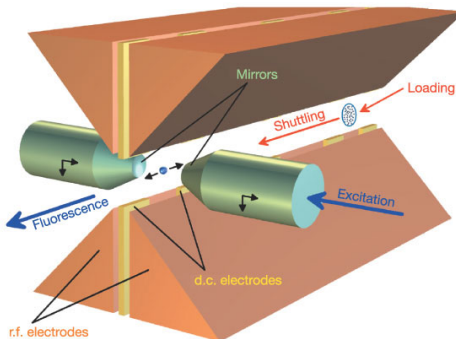


Figure: Apparatus for trapping an ion using r.f. fields. From Myatt et al. **Nature**. 269-273. 2000

Solving the Caldeira-Leggett Model: Analogue of Van-Kampen Modes

- ▶ Guess solutions like $\sim e^{-iut}$.
- ▶ Typically there are no discrete eigenvalues.
- ▶ Instead there are continuum eigenmodes, for each real u :

$$q_u(x) = \mathbf{PV} \frac{Q_u x f(x)}{u^2 - x^2} + C_u Q_u \delta(|u| - x).$$
$$C_u = \frac{u^2 - \Omega_c^2}{\Omega f(|u|)} - \int_{\mathbb{R}} dx \frac{f(|x|)^2}{2(u-x)f(|u|)}.$$

- ▶ Here Q_u is an arbitrary amplitude. The other continuum eigenmodes can be formally solved for using the eigenvalue equations.

Partial Solution of Caldeira-Leggett Through Eigenfunctions

- ▶ The amplitudes $\frac{\partial Q_u}{\partial t} = -iutQ_u$. Gives a solution to Caldeira-Leggett.

$$q(x, t) = \int_{\mathbb{R}} du \frac{Q_u x f(x)}{u^2 - x^2} e^{-iut} + \int_{\mathbb{R}} du C_u Q_u \delta(|u| - x) e^{-iut}$$

$$Q(t) = \int_{\mathbb{R}} du Q_u e^{-iut},$$

$$p(x, t) = - \int_{\mathbb{R}} du \frac{iQ_u u f(x)}{u^2 - x^2} e^{-iut} - \int_{\mathbb{R}} du \frac{i u}{x} C_u Q_u \delta(|u| - x) e^{-iut}$$

$$P(t) = - \int_{\mathbb{R}} du \frac{i u}{\Omega} Q_u e^{-iut},$$

Transformation from Q_u can be Written Using Singular Integral Operators

- ▶ Make the definitions:

$$H[g](v) = \frac{1}{\pi} \int_{\mathbb{R}} dx \frac{g(x)}{x-v}.$$

$$\epsilon_I = \pi f(x)^2 \operatorname{sgn}(x) \quad \text{and}$$

$$\epsilon_R = 2 \frac{x^2 - \Omega_c^2}{\Omega} + \pi H[f(|x|)_-^2].$$

$$T_+[h](u) := \epsilon_R h(|u|) + \epsilon_I H[h(|x|)](u),$$

$$T_-[h](u) := \epsilon_R h(|u|) + \epsilon_I H[\operatorname{sgn}(x)h(|x|)](u),$$

- ▶ Then the previous transformation from Q_u to $(q(x), Q)$ can be written in terms of the symmetric part of Q_u , which is called Q_{u+} :

$$I_c[Q_+] := \left(\frac{1}{f(x)} T_+[Q_{u+}], 2 \int_{\mathbb{R}_+} du Q_{u+} \right)$$

Singular Integral Equations and Riemann-Hilbert Problems

- ▶ Let $h(t)$ be a function defined on \mathbb{R} . Then we define $\phi(z)$ by the Cauchy integral:

$$\phi(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(t) dt}{t - z}$$

- ▶ This function is sectionally analytic in the lower and upper half plane. At the real axis there is a jump discontinuity:

$$\phi_+(t) - \phi_-(t) = h(t) \quad \phi_+(t) + \phi_-(t) = -H[h](t)$$

- ▶ Therefore the function $h(t) + iH[h](t)$ is the boundary value of an analytic function in the upper half plane, and $h(t) - iH[h](t)$ is the boundary value of an analytic function in the lower half plane, for any h .
- ▶ Can write elements of the transformation in this form.

Inverting the Transform Using the Generalized Liouville Theorem

- ▶ Must solve for Q_{u+} :

$$f(x)q(x) = \epsilon_R Q_{u+} + \epsilon_I H[Q_{u+}]$$

- ▶ Subject to the constraints Q_{u+} is symmetric and $2 \int_{\mathbb{R}_+} Q_u du = Q$.
- ▶ Replace each term as a sum of an analytic function in the upper half plane and an analytic function in the lower half plane.
- ▶ Isolate all the upper half plane terms onto the opposite side of all the lower half plane terms.
- ▶ The two sides collectively define an entire function.
- ▶ Apply generalized Liouville theorem: An entire function with a growth rate of z^n at infinity is a polynomial of degree n .
- ▶ Choose the polynomial to satisfy the constraints.

Inverting the Transform II

- ▶ To this write the inverse transformation define:

$$\widehat{T}_+[h](u) := \frac{\epsilon R}{|\epsilon|^2} h(u) - \frac{\epsilon I}{|\epsilon|^2} H[h(|x|)](u),$$

$$\widehat{T}_-[h](u) := \frac{\epsilon R}{|\epsilon|^2} h(u) - \frac{\epsilon I}{|\epsilon|^2} H[\text{sgn}(x)h(|x|)](u).$$

$$Q_{u+} = \widehat{I}_c[q(x), Q] = \widehat{T}_+[f(x)q(x)] + \frac{2u}{\pi\Omega} \frac{\epsilon I}{|\epsilon|^2} Q.$$

$$\bar{Q}(u) = \sqrt{\frac{\pi}{|\epsilon|^2}} \epsilon I Q_{u+} = \sqrt{\frac{\pi}{|\epsilon|^2}} \epsilon I \widehat{T}_+[f(x)q(x)] + \frac{2u}{\Omega} \sqrt{\frac{|\epsilon I|}{\pi|\epsilon|^2}} Q$$

- ▶ Armed with a transformation from $(q(x), Q)$ to $\bar{Q}(u)$, we use a canonical transformation to complete the diagonalization of the Caldeira-Leggett model.

Canonical Transformation

- ▶ Define the type-2 mixed variable generating functional:

$$\mathcal{F}[q, Q, \bar{P}] = \int_{\mathbb{R}_+} \bar{P} \sqrt{\frac{\pi}{|\epsilon|^2}} \epsilon_I \hat{T}[q(x), Q]$$

- ▶ Then the rest of the transformation is:

$$p(x) = \frac{\delta \mathcal{F}}{\delta q} = f(x) \hat{T}_+^\dagger \left[\sqrt{\frac{\pi |\epsilon^2|}{\epsilon_I}} \bar{P} \right]$$

$$P = \frac{\partial \mathcal{F}}{\partial Q} = \int_{\mathbb{R}_+} du \frac{2u \bar{P}}{\Omega} \sqrt{\frac{\epsilon_I}{\pi |\epsilon|^2}}$$

$$\bar{P} = \sqrt{\frac{\pi |\epsilon|^2}{\epsilon_I}} \left(\hat{T}_- [f(x) p(x)] + \frac{2}{\pi} \frac{\epsilon_I}{|\epsilon|^2} P \right)$$

New Hamiltonian: Pure Continuum of Harmonic Oscillators

- ▶ New variables \bar{P} and \bar{Q}

$$\bar{P} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \left(\widehat{T}_-[f(x)p(x)] + \frac{2}{\pi} \frac{\epsilon_I}{|\epsilon|^2} P \right)$$
$$\bar{Q} = \sqrt{\frac{\pi|\epsilon|^2}{\epsilon_I}} \left(\widehat{T}_+[f(x)q(x)] + \frac{2u}{\pi\Omega} \frac{\epsilon_I}{|\epsilon|^2} Q \right) .$$

- ▶ On direct substitution, these variables convert the Hamiltonian of the Caldeira-Leggett model into a pure continuum of harmonic oscillators with Hamiltonian:

$$H[\bar{Q}, \bar{P}] = \int_{\mathbb{R}_+} \frac{du}{2} (\bar{Q}(u)^2 + \bar{P}(u)^2)$$

- ▶ These transformations allow the Caldeira-Leggett model to be solved exactly.

Damping in the Caldeira-Leggett Model

- ▶ Consider an initial condition in the Caldeira-Leggett model.
- ▶ There is a corresponding initial condition $\bar{Q}(u, 0)$, $\bar{P}(u, 0)$.
- ▶ The solution in terms of these variables is:

$$\begin{aligned}\bar{Q}(u, t) &= \bar{Q}(u, 0)\cos(ut) + \bar{P}(u, 0)\sin(ut) \\ \bar{P}(u, t) &= \bar{P}(u, 0)\cos(ut) - \bar{Q}(u, 0)\sin(ut)\end{aligned}$$

- ▶ From this,
$$Q(t) = \int_{\mathbb{R}_+} \frac{2u}{\Omega} \sqrt{\frac{\epsilon_I}{\pi|\epsilon|^2}} (\bar{Q}(u, 0)\cos(ut) + \bar{P}(u, 0)\sin(ut)).$$
- ▶ By the Riemann-Lebesgue lemma this decays to zero with time.
- ▶ Damping is continuum damping.
- ▶ Completely analogous calculation can be made to Landau damping of a plasma. The damping here has the nature of Landau damping.

Linearized Vlasov-Poisson: A Similar Situation

- ▶ Consider the linearized Vlasov-Poisson equation in Fourier space:

$$\frac{\partial f_k}{\partial t} - ikvf_k - \frac{4\pi ie^2}{mk} f'_0(v) \int_{\mathbb{R}} dv f_k = 0.$$

- ▶ There is a transformation, due to Morrison, that allows the exact solution of the Vlasov-Poisson equation:

$$\varepsilon_I(v) = -\frac{4\pi^2 e^2 f'_0}{mk^2 \int_{\mathbb{R}} dv f_0} \qquad \varepsilon_R(v) = 1 + H[\varepsilon_I],$$

$$G_k[f] = \varepsilon_R f + \varepsilon_I H[f] \qquad \widehat{G}_k[f] = \frac{\varepsilon_R}{|\varepsilon|^2} f - \frac{\varepsilon_I}{|\varepsilon|^2} H[f].$$

- ▶ $Q_k(u) = \widehat{G}_k[f_k]$ satisfies $\frac{\partial Q_k(u)}{\partial t} = -iukQ_k(u)$.
- ▶ This makes it possible to identify solutions of Caldeira-Leggett with Vlasov-Poisson and vice-versa.

Plasma Echo: An Inspiration for an Application?

- ▶ In plasmas the density or electric field is typically measured. How do we know that the phase space structure of the distribution function is preserved?
- ▶ Using weakly nonlinear Vlasov theory, two perturbations, separated by time $t = \tau$ of the electric field that have Landau damped may interact and cause the re-appearance of the electric field at some later time t .
- ▶ The plasma echo was predicted and observed in the 1960s.
- ▶ The strength of the echo makes it possible to determine the collisionality.
- ▶ Could it be possible to observe some type of echo effect in the Caldeira-Leggett model?
- ▶ This echo has been suggested in an unpublished arXiv submission by Kuklov, **arXiv:cond-mat/9803351v1** (1998).

Echo Effect in the Caldeira-Leggett Model

- ▶ Relying on the linear theory, and echo may be triggered by driving the bath.
- ▶ Suppose the bath Hamiltonian depends on a parameter:

$$H[q(x), p(x)] = \int \frac{dx}{2} (c(t)q(x)^2 + p(x)^2)$$

- ▶ Let $c(t) = 1 + c\delta(t - \tau)$, which implies that the bath is driven by a strong impulse.
- ▶ At $t = \tau$ the bath coordinates are transformed.
 $q(x, \tau) = q(x, \tau), p(\tau) = p(\tau) - cq(\tau)$.
- ▶ Result is partial time reversal of the evolution:

$$\frac{\partial q(x, \tau)}{\partial t} = xp(x, \tau) - cxq(x, \tau)$$

$$\frac{\partial p(x, \tau)}{\partial t} = -xq(x, \tau) + cx/2(p(x, \tau) - p(x, \tau))$$

- ▶ At time $t = 2\tau$ there is an echo and the initial perturbation to Q is recovered, with magnitude c .

Candidate Systems: Bose-Einstein Condensate

- ▶ Caldeira-Leggett is used for a large number of types of systems, with widely varying types of baths.
- ▶ One intriguing (highly speculative) possibility is the interaction of a Bose-Einstein condensate with a neutral atom trapped in an optical lattice.
- ▶ Confinement of neutrals due to AC Stark effect. Shift of ground state energy. Harmonic potentials possible.
- ▶ Bose-Einstein condensate can coexist with confined neutrals.
- ▶ Study interaction of a neutral (system) with the bath (excitations of the BEC).
- ▶ Drive the Bose-Condensate with a laser to trigger an echo.

Diagram of Optical Lattice

- ▶ There are a large number of possible choices for this configuration, including components for the BEC, the neutrals, the optical lattice, and the method of excitation.

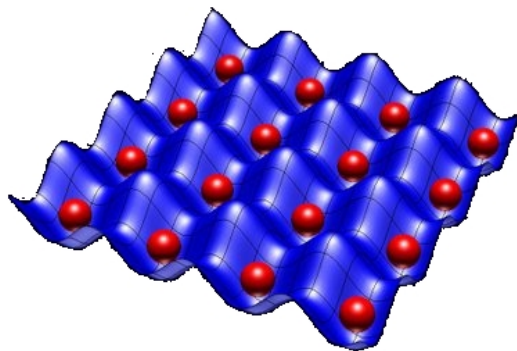


Figure: Optical lattice with trapped neutrals. Imposition of Bose-Einstein condensate could lead to realization of the Caldeira-Leggett model through interaction with Bose-Einstein spectrum. Figure from NIST.

Conclusion

- ▶ Found a transformation that allows for the exact solution of the Caldeira-Leggett model.
- ▶ Use diagonalization to explain the damping in the Caldeira-Leggett model as continuum damping.
- ▶ Established an exact equivalence with the linearized Vlasov-Poisson equation.
- ▶ Suggested the existence of an echo in the Caldeira-Leggett model.
- ▶ Detection and use of echos could lead to a large variety of interesting experiments in systems described by the Caldeira-Leggett model.