The History and Present Status of Gyrokinetic Theory

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Dedicated to J. Bryan Taylor

Thanks to
G. Hammett, W. W. Lee

\(^a\)Work supported by U.S. Dept. of Energy Contract No. DE-AC02-09CH11466.
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www.offthemark.com, cartoon # 1997-06-03.
Gyrokinetics is a method for treating low-frequency fluctuations in magnetized plasmas. Most fundamentally,

\[ \frac{\omega}{\omega_{ci}} = O(\epsilon). \]  

(1)

In the standard ordering,

\[ \frac{\rho_i}{L} = O(\epsilon), \quad \frac{V_E}{c_s} = O(\epsilon). \]  

(2)

Also,

\[ k_\perp \rho_i \ll 1 \quad \text{— guiding-center theory,} \]  

(3a)

\[ k_\perp \rho_i = O(1) \quad \text{— gyrokinetic theory.} \]  

(3b)

Finally,

\[ k_\parallel / k_\perp = O(\epsilon). \]  

(4)

The focus is on the description of gyrocenter motion.
Some recent and forthcoming review articles:

For plasma specialists:

A. J. Brizard & T. S. Hahm,
“Foundations of nonlinear gyrokinetic theory,”
Rev. Mod. Phys. 79, 421 (2007);

X. Garbet, Y. Idomura, L. Villard, & T. H. Watanabe,
“Gyrokinetic simulations of turbulent transport,”
Nucl. Fus. 50, 043002 (2010);

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G. W. Hammett,
“Gyrokinetic turbulence simulations and comparisons with experiments” (in preparation).

For non-plasma physicists:

J. A. Krommes,
“The Gyrokinetic Description of Microturbulence in Magnetized Plasmas,”
Annual Review of Fluid Mechanics
(to be published January, 2012).
The **basic premises** of this talk:

In spite of significant recent concerns, modern nonlinear gyrokinetics is a robust and immensely useful tool for the study of low-frequency effects in magnetized plasmas.
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Along with great power comes great subtlety. GYROKINETICS MUST BE HANDLED WITH CARE!!!
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In spite of significant recent concerns, modern nonlinear gyrokinetics is a robust and immensely useful tool for the study of low-frequency effects in magnetized plasmas.

Along with great power comes great subtlety. GYROKINETICS MUST BE HANDLED WITH CARE!!!

GYROKINETICS IS ONE OF THE MAJOR SUCCESSES OF PLASMA THEORY.
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- Hamiltonian and Lagrangian Approaches (1983–present)
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- Gyrokinetic Field Theory (2000, 2010)
- Recent Topics in Gyrokinetics (last few years)
- Gyrokinetics Under Attack (2008–present)
- Summary and Discussion

*Not discussed:* simulation techniques; collisional effects; physics of zonal flow generation, intrinsic rotation, etc.
Prehistory
(early 1960’s)

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In the beginning, there was the Vlasov equation...

...and one had to deal with the Lorentz force term

\[
\frac{q}{mc} \vec{v} \times \vec{B} \cdot \frac{\partial f}{\partial \vec{v}} = \omega_c \frac{\partial f}{\partial \zeta}.
\]  

(5)

"Of course, we could express our basic set of moment equations ... entirely in terms of the macroscopic velocity \( \vec{u} \) rather than the drift velocity \( \vec{V} \). However, there is no point in doing this since the equations seem to take their simplest form in terms of \( \vec{V} \)."

— Rosenbluth & Simon (1965)

Prescient. But in that same year, ...
When dealing with Braginskii’s equations (moments of the particle kinetic equation), one had to worry about the gyroviscous cancellation.

“We use Braginskii’s results for the ion stress tensor in the case $\Omega_i \tau_i > 1$. The stress tensor consists of three parts. . . The third part, denoted by $\pi_f$, is nonzero even in the absence of collisions and is associated with finite Larmor radius contributions to the momentum transport. We note that this collision-free contribution to the stress tensor and the ion diamagnetic velocity $[\vec{u}_{*i}]$ combine . . . in the following way:

$$m_i n_i \left( \frac{\partial}{\partial t} + \vec{u}_{i,\perp} \cdot \vec{\nabla} \right) \vec{u}_{*i} + (\vec{\nabla}_\perp \cdot \vec{\pi}^{(1)}_f) \perp = \vec{\nabla} \chi, \quad (6)$$

where $\chi = -p_i \hat{z} \cdot \vec{\nabla} \times \vec{u}_i^{(1)} / 2\Omega_i . . .”$

— Hinton & Horton (1971)

$\Rightarrow$ nonlinear polarization drift $\omega_{ci}^{-1} \hat{b} \times (\vec{u}_E \cdot \vec{\nabla}) \vec{u}_E$. 
not $(\vec{u}_E + \vec{u}_{*i}) \cdot \vec{\nabla}$
The theory of adiabatic invariants is crucial.

For a particle gyrating in a circle, the magnetic moment is

\[ \mu = \mu_0 \pm \frac{1}{2} mv^2 / \omega_c. \]  

(7)

In the presence of electric and inhomogeneous magnetic fields, \( \mu_0 \) is no longer conserved.

But a true magnetic moment \( \mu \) is (adiabatically) conserved:

\[ \mu = \mu_0 + \epsilon \mu_1 + \epsilon^2 \mu_2 + \cdots. \]  

(8)

See fundamental work by

- Kruskal (1962),
- Hastie, Taylor, & Haas (1967),
- Taylor (1967).

“… neither here nor elsewhere do we introduce the [gyrocenter] concept directly.”

— Hastie, Taylor, & Haas (1967)
Early Development of Gyrokinetics
(1968–82)

THIS ISN'T EITHER A FAD! - IT'S A MAJOR BREAKTHROUGH!
Early seminal papers on linearized gyrokinetics were by Taylor & Hastie and Rutherford & Frieman.


“... we wish to treat the case where the perpendicular wavelengths may be of the same order as the Larmor radius.”

— Rutherford & Frieman (1968)

Focus was on general geometry.

No clearly articulated concept of the gyrocenter [although \( \mu \) conservation was used and people were worrying about modes with \( k_\perp \rho_i = O(1) \)].
Catto’s 1978 paper on “Linearized gyro-kinetics” was inspirational.

“The present treatment avoids the substantial mathematical complications inherent in [the prior treatments of Taylor & Hastie and Rutherford & Frieman] by introducing the transformation to the guiding center variables and performing the guiding center gyrophase average before specifying the magnetic coordinates to be employed.”

— P. Catto (1978)
The first nonlinear gyrokinetic equation was derived by Frieman & Chen (1982).

E. A. Frieman & L. Chen,
(received 6 October, 1981)

- The nonlinear generalization of Catto’s linearized gyro-kinetics.
- A technical feat: systematic closure of the nonlinear Vlasov equation even in the face of gyroaveraging.
- A publishing breakthrough: ‘gyro-kinetic’ $\Rightarrow$ ‘gyrokinetic.’
- Did not use the full gyrocenter PDF as the dependent variable.
- Their form of the equation unfortunately obscured the role of the polarization drift.

10 days later, . . .
The Genesis of Gyrokinetic Simulation
(1983)
In a seminal paper, Lee (1983) both derived the modern form of the GK-Poisson system and introduced gyrokinetic particle simulation.


“[Catto’s] technique of gyrokinetic change of variables ... provides a starting point for the development of the particle simulation scheme reported here.”

— W. W. Lee (1983)

\[
\frac{\partial \langle F \rangle}{\partial t} + v_\| \nabla_\| \langle F \rangle + \frac{c}{B} \hat{b} \times \vec{\nabla} \langle \Psi \rangle \cdot \vec{\nabla} \langle F \rangle - \frac{q}{m} \nabla_\| \langle \Psi \rangle \frac{\partial \langle F \rangle}{\partial v_\|} = 0, \quad (9)
\]

where \( \langle \Psi \rangle \) is the effective (gyro-averaged) potential:

\[
\langle \Psi \rangle \approx \langle \varphi \rangle + O(\varphi^2). \quad (10)
\]

Here the effective potential is \( \langle \varphi_{\vec{k}} \rangle = J_0(k_\perp v_\perp/\omega_{ci}) \varphi_{\vec{k}}. \)
The gyrokinetic Poisson equation contains an explicit polarization term.

Lee’s gyrokinetic equation evolves the PDF for gyrocenters, but Poisson’s equation involves the charge density of the true particles.

Fig. 3. A time-varying \( \vec{E} \) creates a polarization drift velocity \( \vec{V}^{\text{pol}} \) and the development of polarization charge \( \rho^{\text{pol}} \). **GYROCENTERS DO NOT MOVE WITH THE POLARIZATION DRIFT!**

\[
-\nabla^2 \varphi = 4\pi \rho^G + 4\pi \rho^{\text{pol}},
\]

\[
\approx D_\perp \nabla^2_\perp \varphi
\]

where \( D_\perp \equiv \omega^2_{pi}/\omega^2_{ci} \gg 1 \). **Quasineutrality:** \(-D_\perp \nabla^2_\perp \varphi = 4\pi \rho^G.\)
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Hamiltonian and Lagrangian Approaches (1983–present)
Robert Littlejohn’s influence on the field was seminal.


The basic argument:

- Particle motion is Hamiltonian.
- The Hamiltonian property is preserved under variable transformations.
- Therefore, it should be possible (and it is very desirable) to develop a Hamiltonian theory of gyrocenter motion.
  - use convenient variables
  - conserve phase-space volume
The first self-consistent formulation of Hamiltonian gyrokinetics was given by Dubin, Krommes, Oberman, and Lee (1983).

Key ideas from that paper (constant $\vec{B}$ only):

- Use of noncanonical Poisson brackets,
- Lie transforms ($\tilde{z} = Tz$),
- extended (8D) phase space.

Gyrokinetic equation in 8D Poisson-bracket form:

$$\{ \tilde{F}, \overline{H} \} = 0. \quad (12)$$

Calculation of the gyrocenter Hamiltonian $\overline{H}$ through second order [slab geometry; see recent calculation by Parra & Calvo (2011)]:

$$\overline{H}(\tilde{X}, U, \mu) = \frac{1}{2} m U^2 + \mu \omega_c + q \overline{\Psi} + O(\epsilon^3), \quad (13)$$

where

$$\overline{\Psi} = \langle \varphi \rangle - \frac{1}{2} \omega_{ci}^{-1} \left( \frac{q}{m} \right) \left( \frac{\partial \langle \delta \varphi^2 \rangle}{\partial \mu} + \omega_{ci}^{-1} \langle \nabla \delta \Phi \cdot \hat{b} \times \nabla \delta \varphi \rangle \right) + O(\epsilon^3).$$
More contributions of Dubin et al. . . .

- Definition of the polarization in terms of the variable transformation (now called the ‘pullback’ transformation).
- A gyrokinetic energy invariant.
- The simplest possible derivations of
  - the Hasegawa–Mima equation,
  - the weak-turbulence equations of Sagdeev & Galeev.

But there were some **confusions and imprecise language** as well. . .
CONFUSION #1: What is polarization, really?

\[-\nabla^2 \varphi(\vec{x}, t) = 4\pi \int d\vec{z} \, F(\vec{z}) \delta(\vec{X} + \vec{p} - \vec{x}) + 4\pi \left( \text{stuff } \propto \varphi, \text{ related to } T - 1 \right) \]

Intuitively, most people associate polarization with the presence of an electric field.
CONFUSION #1: What is polarization, really?

\[-\nabla^2 \varphi(\vec{x}, t) = 4\pi \int d\vec{z} F(\vec{z}) \delta(\vec{X} + \vec{p} - \vec{x}) + 4\pi \left( \text{stuff } \propto \varphi, \text{ related to } T - 1 \right) . \]

(15)  

‘polarization’ (???)

Intuitively, most people associate polarization with the presence of an electric field.

“In the absence of external fields, atoms or molecules may or may not have electric dipole moments, but if they do, the moments are randomly oriented. In the presence of a field, the atoms become polarized (or their permanent moments tend to align with the field) and possess on the average a dipole moment. These dipole moments can contribute to the averaged charge density... In the absence of a field there is no average polarization.”

— Jackson (1962)

Except for electrets, which have a permanent electric polarization.

——— 23.2/2 ———
The magnetized plasma is an electret.

Note

\[ \int d\vec{z} \delta(\vec{X} + \vec{p} - \vec{x}) F(\vec{z}) \rightarrow \int d\vec{z} J_0(k_v/c) F \]  \hspace{1cm} \text{(16)}

\[ \text{independent of gyrophase} \]

is not exactly the gyrocenter charge density \( \rho^G \). To have

\[ \nabla \cdot \vec{D} = 4\pi \rho^G \quad \text{('free' charge)} \]  \hspace{1cm} \text{(17)}

with

\[ \vec{D} = \vec{E} + 4\pi \vec{P} \]  \hspace{1cm} \text{(18)}

\((\nabla \cdot \vec{P} = -\rho^{pol})\), one must write

\[ \int d\vec{z} J_0 F = \int d\vec{z} \left[ 1 + (J_0 - 1) \right] F. \]  \hspace{1cm} \text{(19)}

\[ \Delta \rho^{pol} \]

It is not hard to see that \( \Delta \rho^{pol} \) contains diamagnetic flows.
The GK-Poisson system contains (the FLR generalization of) the well-known physics of the particle fluid equations, but it packages that physics differently.

- Gyrokinetic equation: $\langle \vec{E} \rangle \times \vec{B}$ and magnetic drifts.
- Poisson equation: Polarization and diamagnetic drifts.

The fact that the GKE does not contain the diamagnetic drift implies that gyrokinetics takes care of the gyroviscous cancellation automatically.

Unfortunately,

Failure to recognize that the GK Poisson equation properly includes the effects of the diamagnetic drift has been the source of recent confusion.

For example, is a GK conservation law fundamentally different (and not as useful) than one based on the Braginskii equations? NO!
**CONFUSION #2: In practice, how does one truncate the GKE and the Poisson eq’n?**

- **GKE**: $\vec{E} \times \vec{B}$ and magnetic drifts $[O(\epsilon)] + \text{higher-order terms}$.
- **GK Poisson equation**: Polarization arises from gyrocenter-to-particle variable transformation:

$$\text{particle coords} = \text{gyrocenter coords} + O(\epsilon) + \text{higher-order terms}.$$  

*(20)*

The proper way to truncate has been a source of confusion.

Some possibilities:

<table>
<thead>
<tr>
<th></th>
<th>order in $\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>GKE</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>Poisson</strong></td>
<td>1</td>
</tr>
</tbody>
</table>

|       | often used | desired | unbalanced | ??? |

Before one can discuss truncation, one needs a systematic theory of higher-order terms.
Lie transforms provide an efficient way of dealing with higher-order terms.

- Recall

\[ \bar{\mu} = \mu_0 + \epsilon \mu_1 + \epsilon^2 \mu_2 + \cdots. \]  

(21)

- We want to write the GKE in terms of \( \bar{\mu} \):

\[ \frac{\partial F}{\partial t} + \cdots + \frac{\bar{\mu}}{\partial \bar{\mu}} \frac{\partial F}{\partial \bar{\mu}} = 0. \]  

(22)

- It is convenient to use Lie transforms: \( \bar{\mu} = T\mu \), where (Dragt & Finn, 1976)

\[ T = \ldots e^{L_4} e^{L_3} e^{L_2} e^{L_1} \quad [L_w \doteq w_i \partial_i \text{ (on scalars)}]. \]  

(23)

In general, all coordinates must be transformed:

\[ \bar{z}^i = Tz^i; \quad \underbrace{f}_{\text{particles}} = \underbrace{T}_{\text{‘pullback’}} \underbrace{F}_{\text{gyrocenters}}. \]  

(24)
Lagrangian formulations of the single-particle motion allow for great flexibility in the choice of variables.

Lagrange’s variational principle is

$$\delta \int dt \left( \vec{p} \cdot \dot{\vec{q}} - H \right) = 0,$$

or (in the language of differential geometry)

$$\delta \int \gamma = 0,$$

where the fundamental (differential) one-form is

$$\gamma = \vec{p} \cdot d\vec{q} - H dt.$$ 

The variational principle is indifferent to the choice of variables:

$$\gamma = \gamma_i dz^i = \bar{\gamma}_i d\bar{\zeta}^i.$$ 

How does one choose a set of new variables that includes a conserved $\mu$?
The key idea: Use Noether’s theorem.

Noether’s theorem (roughly):

Any symmetry of the Lagrangian is associated with a conservation law.

Noether’s theorem a la Littlejohn (1983) and especially Cary & Littlejohn (1983): If the $\gamma_i$’s are independent of the gyrophase angle $\zeta$, then $\gamma_\zeta \equiv \mu$ is conserved.

Therefore, the basic program is to construct (perturbatively) a new one-form $\gamma$ whose coefficients are independent of $\zeta$:

$$\tilde{\gamma} = T^{-1}\gamma + dS.$$  (28)

When $\gamma$ is written in terms of the guiding-center variables $\{\tilde{X}, U, \mu_0, \zeta\}$, gyrophase dependence enters through the potentials:

$$\begin{align*} 
\tilde{A}(\tilde{X} + \tilde{\rho}(\zeta)), & \quad \varphi(\tilde{X} + \tilde{\rho}(\zeta)) \quad \text{Assume } \rho_i/L = O(\epsilon). \end{align*}$$

background $\tilde{B}_0 [O(\epsilon^{-1})]$ & turbulence $[O(\epsilon)]$
The basic program can be easily carried out through the first few orders (Hahm, 1988).

The basic electrostatic gyrokinetic one-form (Hahm, 1988):

\[
\gamma = \overline{A(\vec{X})} \cdot d\vec{X} + \frac{\mu}{\epsilon^{-1}} d\zeta - \overline{H} \, dt,
\]

(29)

where

\[
\overline{H} = \frac{1}{2} m U^2 + \overline{\mu} \omega_c + q \overline{\langle \phi \rangle} + H_2 + O(\epsilon^3).
\]

(30)

\[
H_0 = O(1) \quad H_1 = O(\epsilon)
\]
The basic program can be easily carried out through the first few orders \((\text{Hahm, 1988})\).

The basic electrostatic gyrokinetic one-form \((\text{Hahm, 1988})\):

\[
\mathbf{\gamma} = \overbrace{\mathbf{A}(\mathbf{X}) \cdot d\mathbf{X}}^{\epsilon^{-1}} + \overbrace{\mu d\zeta}^{O(1)} - \overbrace{H dt}^{O(1)},
\]

(29)

where

\[
H = \frac{1}{2}m\overline{U}^2 + \overline{\mu}\omega_c + q\langle \varphi \rangle + H_2 + O(\epsilon^3).
\]

(30)

\[
H_0 = O(1), \quad H_1 = O(\epsilon).
\]

Just how high must one go in order to include all relevant physics?
The basic program can be easily carried out through the first few orders (Hahm, 1988).

The basic electrostatic gyrokinetic one-form (Hahm, 1988):

\[ \gamma = \frac{\mathbf{A}(\vec{X}) \cdot d\vec{X}}{\epsilon^{-1}} + \mu d\zeta - \mathcal{H} \, dt, \]  

\[ \epsilon^{-1} \quad O(1) \]  

where

\[ \mathcal{H} = \frac{1}{2} m \bar{U}^2 + \bar{\mu} \omega_c + q \langle \varphi \rangle + H_2 + O(\epsilon^3). \]  

\[ H_0 = O(1) \quad H_1 = O(\epsilon) \]  

Just how high must one go in order to include all relevant physics?

Recently it has been asserted that one may need to calculate \( H_3 \) in order to properly deal with issues relating to momentum conservation and toroidal rotation!!!
Gyrokinetic Noise
Ion polarization shielding reduces noise due to discrete gyrocenters.

In a weakly coupled, thermal-equilibrium many-body plasma,

\[
\frac{\langle \delta E^2 \rangle (\vec{k})}{8\pi} = \frac{T/2}{1 + k^2 \lambda_D^2}.
\]  

(31)

(This is a consequence of the fluctuation–dissipation theorem.)

The analogous result for a gas of discrete gyrocenters was obtained by Krommes, Lee, and Oberman (1986):

- A nontrivial formula, but with simple limits.
- Fluctuations are significantly reduced by ion polarization.
- This is relevant to particle simulation, which is basically a Monte Carlo sampling technique (Aydemir, 1994; Hu & Krommes, 1994).
Lower noise may be achieved by writing $\delta F = F - F_0$ and treating $F_0$ as a smooth, known function.

The basic ideas were spawned by such pioneers as

- Beyers (early 1970’s) (pre-gyrokinetic),
- Tajima & Perkins (1983),
- Kotschenreuther ($\lesssim$ 1991),
- Dimits & Lee (1993),
- Parker & Lee (1993).

The method involves the introduction of the particle weight

$$w = \delta F/F.$$  \hfill (32)

An attempt at an analytical calculation of $\delta F$ noise was made by Hu & Krommes, 1993.

Much later, Nevins, Hammett, Dimits, Dorland, and Shumaker (2005) discussed “Discrete particle noise in particle-in-cell simulations of plasma microturbulence.” (Some PIC simulations may be noise-dominated.)
Gyrokinetic Codes
(1983–present)

```c
#include <stdio.h>
int main(void)
{
    int count;
    for (count = 1; count <= 500; count++)
        printf("I will not throw paper airplanes in class.");
    return 0;
}
```
Gyrokinetic codes have been very successful.

- Particle-in-cell (PIC):
  - GTC — Lin et al. (1998)
  - GEM — Chen & Parker (2003)
  - GTS — Wang et al. (2006)
  - ORB5 — Jolliet et al. (2007)
  - XGC1 — Chang et al. (2009)

- Continuum (Vlasov)
  - GS2 — Dorland et al. (2000) [based on the linear code of Kotschenreuther et al. (1995); see also the AstroGK code of Numata et al. (2010)]
  - GENE — Jenko et al. (2000)
  - GYRO — Candy & Waltz (2003)
  - GT5D — Idomura et al. (2008)

- Hybrid (semi-Lagrangian)
  - GYSELA — Grandgirard et al. (2006)
FIG. 4: Full-torus (‘global’) GENE simulation of a discharge in the TCV tokamak that exhibits an internal transport barrier (ITB). Realistic input data and comprehensive physics are used. The figure displays contours of electrostatic potential (stream function). In this cross section, the ITB corresponds to a fairly narrow ring near the mid minor radius of the torus. From Görler et al., 2011.
FIG. 5: Predictions (red curves) of the TGYRO code (https://fusion.gat.com/theory/Tgyrooverview) for DIII-D discharge 128913 compared with experimental measurements (discrete data points). The electron temperature is particularly well-reproduced. 10 simulation radii (10 instances of GYRO) were used. All profiles were evolved by TGYRO using nonlinear GYRO calculations. Courtesy of J. Candy.
FIG. 6: Gyrokinetic simulation of ITG turbulence by the PIC code XGC1 in an edge pedestal region with the realistic diverted geometry of the DIII-D device. From Chang et al. (2009).
FIG. 7: One-dimensional energy spectra of the perpendicular magnetic (solid), electric (dashed), and parallel magnetic (dot-dashed) fields from gyrokinetic simulations (with AstroGK) of kinetic Alfvén wave (KAW) turbulence from the ion to electron Larmor radius scales. These spectra demonstrate that KAW turbulence can indeed yield energy spectra reaching the electron scales, as found in recent observations. From Howes et al. (2011).
Gyrokinetic Field Theory
(2000, 2010)

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Sugama (2000) and Brizard (2000) showed that GKs is derivable from a variational formulation. Basically and a bit schematically,

\[ S[F, A_\mu] = S_{\text{EM}} - \int d\vec{z} \bar{F} \bar{H}. \]  

(33)

It can be shown that

\[ \frac{\delta S}{\delta F} \Rightarrow \text{GKE}, \]  

(34a)

\[ \frac{\delta S}{\delta A_\mu} \Rightarrow \text{gyrokinetic Maxwell equations}. \]  

(34b)

So in particular, with \( \varphi = A_0 \),

\[ \frac{\delta S}{\delta \varphi} \approx -\int d\vec{z} \bar{F} \frac{\delta \bar{H}}{\delta \varphi} \Rightarrow \text{gyrokinetic Poisson equation}. \]  

(35)

The variational formulation figures prominently in recent derivations of the conservation law for gyrokinetic toroidal angular momentum.

The variational formulation supercedes the technique of truncating the GKE and the GK Poisson equation.

A basic principle (strongly advocated by B. Scott):

**Make all approximations on the Hamiltonian!**

Then *take what you get.*

For example, let (schematically)

$$H = H_0 + \epsilon \varphi + \frac{1}{2} \epsilon^2 \varphi^2 + \cdots.$$  \hfill (36)

Then the kinetic equation is

$$\partial_t F + \{ F, H_0 \} + \epsilon \varphi + \frac{1}{2} \epsilon^2 \varphi^2 + \cdots = 0$$ \hfill (37)

Then the GK Poisson equation is

$$0 = \int d\vec{z} F \left( \frac{\delta H}{\delta (\epsilon \varphi)} \right) = \int d\vec{z} F \left( \frac{1}{\epsilon \varphi} \right) + \epsilon \varphi + \cdots.$$ \hfill (38)
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Recent Topics in Gyrokinetics
(last few years)
Gyrokinetic theory has not been stagnant:

Some topics of current interest:

- Saturation of plasma turbulence by coupling to damped gyrokinetic eigenmodes (Terry & Hatch).
- Absolute GK statistical equilibria (Zhu & Hammett).
- Entropy, phase-space cascades, and dissipation (Plunk, Schekochihin, etc.)
- Submarginal turbulence (Oxford group).
Turbulence may saturate by coupling to damped eigenmodes.

FIG. 8: Gyrokinetic eigenmode spectrum for ITG fluctuations, demonstrating the possibility of nonlinear coupling of an unstable ITG mode (red) to damped eigenmodes without spectral cascade. From Hatch et al. (2011).
Absolute gyrokinetic statistical equilibria can give useful insights about nonequilibrium turbulence.

It is well known that nondissipative (Euler) fluid equations truncated in Fourier space possess a Liouville theorem and absolute statistical equilibria:

\[ P[\varphi_{\vec{k}}] \propto \exp \left( - \sum_{i=1}^{N} \alpha_i I_i[\varphi_{\vec{k}}] \right), \tag{39} \]

where the \( I_i \) are quadratic invariants.
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\[ P[\varphi_k] \propto \exp \left( - \sum_{i=1}^{N} \alpha_i I_i[\varphi_k] \right) \tag{39} \]

where the \( I_i \) are quadratic invariants.

Gyrokinetics adds velocity degrees of freedom. There can be either
- many more (entropy-related) invariants (2D), or
- modified invariants (3D).

Zhu & Hammett (2010) have discussed gyrokinetic absolute statistical equilibria.

Possible significant modifications to nonequilibrium cascades.

Amazing contact to earlier work on GK noise.

There is further research to be done...
The theory of **GK entropy cascades** is relevant to the physics of dissipation and turbulence saturation.

**Background:** The entropy paradox in collisionless PIC simulation.

**FIG. 9:** Highly schematic representation of the entropy paradox in collisionless $\delta F$ PIC simulation. The turbulent flux nicely saturates, but the mean-square weight, $\langle w^2 \rangle \sim \langle \delta F^2 \rangle$, keeps increasing with time. **Paradox!**

- The entropy paradox is resolved by the inclusion of *collisional dissipation* (Krommes & Hu, 1994).
‘Nonlinear phase mixing’ provides an efficient route to GK entropy cascade.

FIG. 10: Illustration of nonlinear phase mixing. Contours of constant $\varphi$ are plotted; overlaid are the orbits of two particles that have the same gyrocenter. During a nonlinear turnover time, phase-mixing will decorrelate the motion because their gyroradii differ by the correlation length of the turbulence. From Plunk et al. (2010).
Detailed theory of the entropy cascade has been verified by numerical GK simulation.

Detailed predications have been made for the spectral cascades resulting from nonlinear phase mixing and verified numerically in 2D (Tatsuno et al., 2009) and 3D (Navarro et al., 2011).

**FIG. 11:** GENE simulation of 3D ITG turbulence; from Navarro et al. (2011).

One expects to learn a lot about the physics of saturation of GK turbulence by pursuing these issues. *More to come...*
GK simulations have been used to map out the parameter space of heat flux vs. $\nabla u$ and $\nabla T$.

\[ \frac{Q}{Q_{GB}} \]

FIG. 12: GK simulations of involving both flow shear and temperature gradient. Left: linear growth rates; right: heat flux. From Barnes et al., 2010.

- For some parameters, the turbulence can be submarginal ($\gamma^{\text{lin}} < 0$).
- This is a characteristic feature of non-normal systems ($L L^\dagger \neq L^\dagger L$).
- The theoretical description of submarginal and/or non-normal GK systems is a challenging topic of current interest.
- There are interesting connections to the physics of zonal-flow generation, stochastic structural stability, etc.
Gyrokinetics Under Attack
(2008–present)
Recently significant concerns have been raised about the gyrokinetic formalism.

There are several issues, but time is short...
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One assertion:

The standard (truncated) GK-Poisson system is unsuitable for discussion of momentum transport on long time scales because it is unable to properly calculate the radial electric field.

— F. Parra & P. Catto (2008–10)
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— F. Parra & P. Catto (2008–10)

This is a significant, well-motivated concern raised by thoughtful, expert researchers. We must take it seriously. There are deep issues here that need to be understood.
This problem has generated more than ten papers and hundreds of pages of text over the last few years. It involves a collision of the old (pre-Hamiltonian) and the new (Hamiltonian, Lagrangian, variational) points of view, thus major communication problems. Nevertheless, substantial progress has been made toward resolving the various points of view and defining the real issues.

The basic assumptions:

- **Low-flow ordering:** \( \langle u_\phi \rangle / c_s = O(\epsilon) \).

- **Gyro-Bohm transport:** \( \mu \sim \left( \frac{\rho_s}{L} \right) \left( \frac{cT_e}{eB} \right) = \left( \frac{\rho_s}{L} \right) \rho_sc_s \).

Then

\[
\frac{L \nabla \Pi}{nm_i c_s^2} \sim L \frac{\mu \nabla^2 \langle u \rangle}{c_s^2} = \left( \frac{\rho_s}{L} \right)^2 \left( \frac{\langle u \rangle}{c_s} \right) = O(\epsilon^3).
\]  

(40)
We now come to the basic question and concerns:

Can one use the standard GK-Poisson formulation to calculate momentum transport, toroidal rotation, etc., on long time scales?

Originally directed at ‘full-$F$’ codes, this question has relevance to $\delta F$ codes as well.

In essence, the basic assertions are:

- Standard gyrokinetics is truncated to too low an order to represent momentum conservation correctly. (Because of ‘intrinsic ambipolarity’ through second order, it is difficult to calculate the long-wavelength radial electric field correctly.)

- To get a correct description of momentum conservation from the standard GK-Poisson approach, one must keep very small terms (including the third-order Hamiltonian $H_3$).

- Discussing gyrokinetic momentum conservation is pointless because what is really required is a conservation law in laboratory coordinates. (This is really a fundamental criticism of the entire gyrokinetic formalism.)
For axisymmetric systems, Scott & Smirnov (2010) have derived the exact form of the GK conservation law for toroidal angular momentum.

B. Scott & J. Smirnov,
“Energetic consistency and momentum conservation in the gyrokinetic description of tokamak plasmas,”

In my opinion, this is the most important and substantive paper on basic gyrokinetic theory in recent times.

It refocused the discussion of momentum-conservation issues in a major way.

It has practical implications as well, e.g. for the construction of momentum-conserving codes.

The conservation law of Scott & Smirnov has recently been rederived more concisely and generally by Brizard (2010) and Brizard & Tronko (2011). But SS were first, and they deserve considerable credit for their deep insights.
For axisymmetric systems, Scott & Smirnov (2010) have derived the exact form of the GK conservation law for toroidal angular momentum.

- Begin with the (covariant) component of the canonical momentum for the gyrocenter, \( P_\phi = -(q/c)\psi + mv_\parallel b_\phi \).

- Average \( P_\phi \) over the gyrocenter PDF and take the time derivative.

- Replace \( \partial_t F \) with the right-hand side of the GKE.

- Also use the quasineutrality condition to effect a major cancellation. (Large terms involving \( \psi \) disappear ⇒ save one order!)

- Left with one term, \( \langle \int F \frac{\partial H}{\partial \phi} \rangle \), which must be reduced further. (Does not obviously look like a divergence.)

- Prove a nontrivial theorem about the relation between \( \partial / \partial \phi \) and a functional derivative.

- Introduce the representation of the quasineutrality condition as the functional derivative of an action principle. Prove that all nondivergence terms cancel!
One now has the momentum conservation law valid for an arbitrary Hamiltonian.

\[
\partial \left( \mathcal{P}_\parallel + c^{-1} \mathcal{P}_\psi \right) + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left( \Pi_\parallel + \Pi_\perp \right) = 0. \tag{41}
\]
One now has the momentum conservation law valid for an arbitrary Hamiltonian.

\[ \frac{\partial (P_{\phi\parallel} + c^{-1} P^\psi)}{\partial t} + \frac{1}{V'} \frac{\partial}{\partial \psi} V' (\Pi_{\phi\parallel} + \Pi_{\phi\perp}) = 0. \]  

(41)

**FIG. 13:** Important vectors.

Total toroidal flow = sum of projections of (i) parallel gyro-center flow, (ii) \( \vec{E} \times \vec{B} \) flow due to radial polarization field, and (iii) diamagnetic flow.
One now has the momentum conservation law valid for an arbitrary Hamiltonian.

\[
\frac{\partial (P_{\phi||} + c^{-1} P^\psi)}{\partial t} + \frac{1}{V'} \frac{\partial}{\partial \psi} V' (\Pi_{\phi||} + \Pi_{\phi\perp}) = 0. \quad (41)
\]

Here the pieces of the toroidal angular momentum density are

\[
P_{\phi||} \doteq \sum_s (\pi m)_s \left\langle \int \hat{F} F_s v_{||} b_{\phi} \right\rangle, \quad (42a)
\]

\[
P^\psi \doteq - \sum_s \pi_s \left\langle \int \hat{F} \left( \frac{\partial H}{\partial \hat{E}} \right)^\psi \right\rangle + \cdots, \quad (42b)
\]

and the momentum fluxes are

\[
\Pi_{\phi||} \doteq \sum_s (\pi m)_s \left\langle \int \hat{F} \left( \frac{\partial H}{\partial \hat{E}} \right)^\psi v_{||} b_{\phi} \right\rangle, \quad (43a)
\]

\[
\Pi_{\phi\perp} \doteq \sum_s \pi_s \left\langle \int \hat{F} \left( \frac{\partial H}{\partial \hat{E}} \right)^\psi \frac{\partial \phi}{\partial \phi} \right\rangle + \cdots \quad (43b)
\]

unfamiliar corrections to the basic Reynolds stress.
How big is the perpendicular Reynolds stress?

We have

$$\Pi_{\psi \phi} \equiv \Pi \equiv \sum_s n_s \left\langle \int F \left( \frac{\partial H}{\partial \mathbf{E}} \right) \psi \frac{\partial \phi}{\partial \mathbf{F}} \right\rangle + \cdots. \quad (44)$$

The contribution from $H_2$ is nominally of second order:

$$\Pi^{[2]} \sim (R \nabla \psi) nm_i \langle V_{E,x} V_{E,z} \rangle = O(\epsilon^2). \quad (45)$$

- Recall that we need $\Pi = O(\epsilon^3)$ for gyro-Bohm transport.
- But $\Pi^{[2]}$ is an off-diagonal component.
- Microscopic symmetry of the background turbulence should make $\Pi^{[2]}$ vanish in the absence of symmetry breaking.
- Conventional arguments would give

$$\Pi^{[2]} \sim -\mu \partial_x \langle u \rangle + (\text{pinch}) + \left( \text{residual stress} \right) = O(\epsilon^3). \quad (46)$$
How big is the perpendicular Reynolds stress?

We have

$$\Pi_{\phi \perp} = \Pi \equiv \sum_s \overline{n}_s \left\langle \int \vec{P} \cdot F \left( \frac{\partial H}{\partial \vec{E}} \right)^\psi \frac{\partial \phi}{\partial \phi} \right\rangle + \cdots. \quad (44)$$

The contribution from $H_2$ is nominally of second order:

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$$\Pi^{[2]} \sim -\mu \partial_x \langle u \rangle + (\text{pinch}) + \left( \text{residual stress} \right) = O(\epsilon^3). \quad (46)$$

- But now we must worry about the possibility that $\Pi^{[3]}$ is also important. Is $\Pi^{[3]}$ also $O(\epsilon^3)$?
What is the form of $H_3$?

To calculate $H^{(3)}$, see Parra & Catto (2010) or use the Lagrangian one-form method (Mishchenko & Brizard, 2011; Krommes, 2011). From

$$\gamma = T^{-1} \gamma + dS$$

(47)

and the expansion of

$$T^{-1} = e^{-L_1} e^{-L_2} \ldots,$$

(48)

one has

$$\gamma^{(-1)} = \gamma^{(-1)} + dS^{(-1)},$$

(49a)

$$\gamma^{(0)} = \gamma^{(0)} - L_1 \gamma^{(-1)} + dS^{(0)},$$

(49b)

$$\gamma^{(1)} = \gamma^{(1)} - L_1 \gamma^{(0)} + \left( -L_2 + \frac{1}{2} L_1^2 \right) \gamma^{(-1)},$$

(49c)

$$\gamma^{(2)} = \gamma^{(2)} - L_1 \gamma^{(1)} + \left( -L_2 + \frac{1}{2} L_1^2 \right) \gamma^{(0)} + \left( -L_3 + L_1 L_2 - \frac{1}{6} L_1^3 \right) \gamma^{(-1)},$$

(49d)

$$\gamma^{(3)} = \gamma^{(3)} - L_1 \gamma^{(2)} + \left( -L_2 + \frac{1}{2} L_1^2 \right) \gamma^{(1)} + \left( -L_3 + L_1 L_2 - \frac{1}{6} L_1^3 \right) \gamma^{(0)}$$

$$+ \left( -L_4 + L_1 L_3 + \frac{1}{2} L_2^2 - \frac{1}{2} L_1^2 L_2 - \frac{1}{24} L_1^4 \right) \gamma^{(-1)}.$$  

(49e)
The cold-ion limit is a good test case.

In the cold-ion limit \((T_i \to 0)\)...

- \(H_2\) arises by averaging two powers of the gyroradius vector:
  \[
  H_2 \sim \langle \vec{\rho} \cdot \vec{\rho} \rangle : \nabla_\perp \nabla_\perp \phi \sim |\nabla_\perp \phi|^2. \tag{50}
  \]

- \(H_3\) arises by averaging four powers of \(\vec{\rho}\):
  \[
  H_3 \sim \langle \vec{\rho} \cdot \vec{\rho} \cdot \vec{\rho} \cdot \vec{\rho} \rangle :: \nabla_\perp \nabla_\perp \nabla_\perp \nabla_\perp \phi. \tag{51}
  \]

- A representative term (constant \(\vec{B}\)):
  \[
  H_3 \sim |\nabla_\perp \phi|^2 \nabla^2_\perp \phi. \tag{52}
  \]

[One could have guessed this (and did).] A recent result:
The cold-ion limit is a good test case.

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- A representative term (constant \(\vec{B}\)):
  \[
  H_3 \sim |\vec{\nabla}_\perp \varphi|^2 \nabla^2_\perp \varphi. \tag{52}
  \]

[One could have guessed this (and did).] A recent result: For \(T_i \to 0\), the contributions of \(H_3\) to the perpendicular Reynolds stress are actually \(O(\epsilon^4)\), hence negligible (Krommes, 2011).

Unfortunately, that’s not the end of the story...
Parra & Calvo (2011) find that the first FLR corrections remain $O(\epsilon^3)$ [really $O(\epsilon^3 (k_\perp \rho_i)^2)$].

Indeed, if in fact they were $O(\epsilon^4)$, an amazing theorem would have to hold.

What is the physical meaning of the third-order terms???

Some kind of contribution to intrinsic rotation?
FLR corrections may contribute!

- Parra & Calvo (2011) find that the first FLR corrections remain $O(\epsilon^3)$ [really $O(\epsilon^3 (k_{\perp} \rho_i)^2)$].
- Indeed, if in fact they were $O(\epsilon^4)$, an amazing theorem would have to hold.
- What is the physical meaning of the third-order terms???
  Some kind of contribution to intrinsic rotation?

  *This is the way the world ends*
  *This is the way the world ends*
  *This is the way the world ends*
  *Not with a bang but a whimper.*

  — T. S. Eliot, *The Hollow Men* (1925)
FLR corrections may contribute!

- Parra & Calvo (2011) find that the first FLR corrections remain $O(\epsilon^3)$ [really $O(\epsilon^3 (k_\perp \rho_i)^2)$].
- Indeed, if in fact they were $O(\epsilon^4)$, an amazing theorem would have to hold.
- What is the physical meaning of the third-order terms??
  Some kind of contribution to intrinsic rotation?

- It seems clear that macroscopic symmetry breaking of the second-order stresses is important (dominant?).
- Gyrokinetics will be crucial in both the analytical theory and numerical simulation of the physics effects contributing to intrinsic rotation. (See yesterday’s poster by Parra, and today’s talk by Weixing Wang.)
Even if $H_3$ contributes to momentum flux, it is not necessarily required in the GKE.

At most, we need $\Pi \sim \langle \varphi^3 \rangle$. Now the structure of the GKE is

$$\frac{\partial_t F}{\partial_t \varphi} + H_1 F + H_2 F + H_3 F = 0.$$  \hfill (53)

Note that even $H_1$ generates fluctuations: $\partial_t \varphi + \varphi^2 = 0$. And of course $\langle \varphi_1^3 \rangle = O(\epsilon^3)$.

There is a useful analogy to weak-turbulence theory. Need to work as hard as $\varphi_3$, but don’t need a cubic nonlinearity to generate $\varphi_3$; rather, $\varphi_3 = \varphi_1 \varphi_2 = \varphi_1^3$.

By pursuing this line of reasoning,

$F = \text{full-} F$: (maybe) need $H_3$ to get all fluxes quantitatively correct

(but still nontrivial physics effects with just $H_2$).

$F \rightarrow \delta F$ (+ separate fluid momentum equation):

OK with $H_2$ (or maybe even just $H_1$).

(Must be prepared to evaluate $\Pi$ sufficiently accurately.)
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Summary and Discussion
**Timeline (the early years)**

- **Rosenbluth & Simon (1965)** — moment equations are simplest in terms of the drift velocity
- **Rutherford & Frieman (1968); Taylor & Hastie (1968)** — linearized GKS in general geometry
- **Hinton & Horton (1971)** — gyroviscous cancellation
- **Catto (1978)** — do transformation to guiding-center variables first!
- **Littlejohn (1979–82)** — noncanonical Hamiltonian techniques
- **Frieman & Chen (1982)** — first nonlinear GKE
- **Lee (1983)** — modern form of the GK-Poisson system; GK particle simulation
- **Dubin, Krommes, Oberman, & Lee (1983)** — self-consistent Hamiltonian GKS
- **Littlejohn (1983) and Cary & Littlejohn (1983)** — Lagrangian methods; Noether’s theorem
- **Krommes, Lee, & Oberman (1986)** — GK noise
- **Hahm (1988)** — GKS via the one-form method
Timeline (the modern era)

Please note: This timeline may not fully reflect the talk because in the end I decided to not use these slides.

- Sugama (2000); Brizard (2000) — variational principle; gyrokinetic field theory
- Parra (2008) — dissertation on GK momentum conservation
- Schekochihin et al. (2009) — astrophysical GKS
- Plunk et al. (2010) — GK entropy cascade
- Zhu & Hammett (2010) — absolute GK statistical equilibria
- Scott & Smirnov (2010) — GK conservation law for toroidal angular momentum
Gyrokinetics is the workhorse for modern research on low-frequency microturbulence in magnetized plasmas. It is a very powerful tool that has been used successfully in many different applications in both fusion and astrophysics.

- Analytical theory of drift waves and related modes.
- Statistical mechanics of magnetized systems.
- Simulations of turbulent transport.
- Interpretation of solar-wind data.
- Theory of momentum transport
- Simulations of intrinsic rotation.

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- Simulations of intrinsic rotation.

...


Barnes et al. (2011). To be submitted.


value codes for kinetic toroidal plasma instabilities.


